ELEN E3401: Electromagnetics

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Lecture #21

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Plane-wave propagation in lossy media

Apply to general – linear, isotropic, homogeneous media

$$\gamma = \alpha + j\beta \qquad \epsilon' = \epsilon \qquad \epsilon'' = \frac{\sigma}{\omega}$$

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]\right\}^{1/2}$$

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]\right\}^{1/2}$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\,\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

Plane-wave propagation in lossy media

1. Perfect dielectric: $(\sigma = 0)$ → reduce to lossless case

$$\alpha = 0, \beta = k = \omega\sqrt{\mu\epsilon} \qquad \eta_c = \eta$$

2. Lossy medium: $\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon}$

(a) Low Loss: $\frac{\epsilon''}{\epsilon'} \ll 1 \qquad \frac{\epsilon''}{\epsilon'} < \frac{1}{100}$

(b) Good conductor: $\frac{\epsilon''}{\epsilon'} \gg 1 \qquad \frac{\epsilon''}{\epsilon'} > 100$

(c) Quasi conductor: $\frac{1}{100} \le \frac{\epsilon''}{\epsilon'} \le 100$

Low-loss dielectric

$$\gamma^2 = -\omega^2\mu(\epsilon' - j\epsilon'') \quad \Rightarrow \quad \gamma = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{\frac{1}{2}}$$

Using binomial series we can approximate $(1-x)^{1/2} \approx 1 - \frac{x}{2}$ (for $|x| \ll 1$)

Low loss dielectric $\left|j\frac{\epsilon''}{\epsilon'}\right| \ll 1$:

$$\gamma \approx j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{2\epsilon'}\right)$$

$$\alpha \approx \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \quad \text{[Np/m]}$$

$$\beta \approx \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu\epsilon} \quad \text{[rad/m]}$$

(same as k for lossless)

Low-loss dielectric

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

again apply binomial expansion: $(1-x)^{-1/2} \approx 1 + \frac{x}{2}$

$$\eta_c \approx \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right) = \sqrt{\frac{\mu}{\epsilon}}\left(1 + j\frac{\sigma}{2\omega\epsilon}\right)$$

Since $\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} < \frac{1}{100} \rightarrow$ can ignore

$$\eta_c \approx \sqrt{\frac{\mu}{\epsilon}}$$

Good Conductor

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}-1\right]\right\}^{1/2} \qquad \beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}+1\right]\right\}^{1/2}$$

For $\frac{\epsilon''}{\epsilon'} > 100$ can approximate:

$$\alpha \approx \omega\sqrt{\frac{\mu\epsilon''}{2}} = \omega\sqrt{\frac{\mu\sigma}{2\omega}} = \sqrt{\pi f\mu\sigma}\ [Np/m]$$

$$\beta = \alpha \approx \sqrt{\pi f\mu\sigma}\ [rad/m]$$

Good Conductor

$$\eta_c = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

$$\frac{\epsilon''}{\epsilon'} > 100$$

recall: $\sqrt{j} = \frac{1+j}{\sqrt{2}}$

$$\eta_c \approx \sqrt{\frac{j\mu}{\epsilon''}} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma}$$

For perfect conductor → $\sigma = \infty$, $\alpha = \beta = \infty$ and $\eta_c = 0$

equivalent to short circuit in transmission line

Summary of propagation in materials

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right]\right]^{1/2}$	0	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu\sigma}$	(Np/m)
$\beta =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}+1\right]\right]^{1/2}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu\sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$(1+j)\frac{\alpha}{\sigma}$	(Ω)
$u_p =$	ω/β	$1/\sqrt{\mu\varepsilon}$	$1/\sqrt{\mu\varepsilon}$	$\sqrt{4\pi f/\mu\sigma}$	(m/s)
$\lambda =$	$2\pi/\beta = u_p/f$	u_p/f	u_p/f	u_p/f	(m)

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$.

Example: copper

Copper has:

$$\mu = \mu_0 = 4\pi \times 10^{-7}\ \left[\frac{H}{m}\right], \epsilon = \epsilon_0 = \frac{1}{36\pi} \times 10^{-9}\ \left[\frac{F}{m}\right], \sigma = 5.8 \times 10^{7}\ \left[\frac{S}{m}\right]$$

Assuming parameters do not change with frequency, over what spectral range is copper a good conductor?

Good conductor: $$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} > 100$$

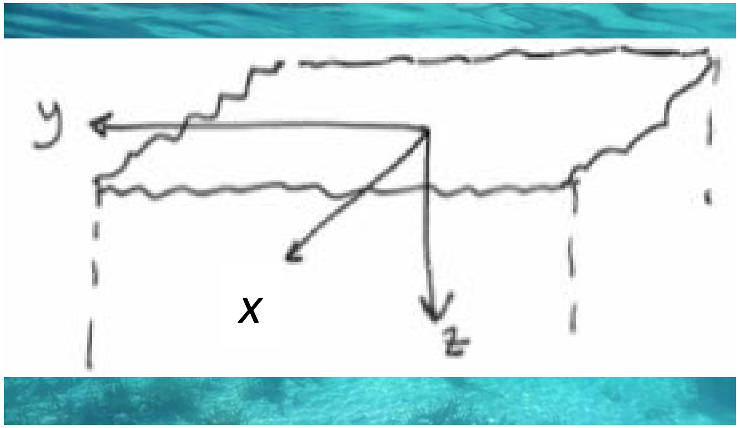
$$\omega = 2\pi \mathrm{f} < \frac{\sigma}{100\epsilon}$$

$$\mathrm{f} < \frac{\sigma}{200\pi\epsilon} = \frac{5.8 \times 10^{7}}{(200\pi)\left(\frac{1}{36\pi} \times 10^{-9}\right)} = 1.04 \times 10^{16}\ Hz$$

As long as f < ~10^{16} Hz, copper good conduction (UV light)

Example: plane wave in seawater

Consider a uniform plane wave in seawater



Plane wave in x-y

Propagate in $+z$

$$\epsilon_r = 80, \mu_r = 1, \sigma = 4\ S/m$$

At z = 0, $\vec{H}(0,t) = \hat{y}100\cos(2\pi 10^3 t + 15°)\ \ [mA/m]$

a) Obtain $\vec{E}(z,t)$ and $\vec{H}(z,t)$

b) Obtain depth where magnitude of $\vec{E}$ is 1% of z=0

Example: plane wave in seawater

a) Obtain $\vec{E}(z,t)$ and $\vec{H}(z,t)$

Since $\vec{H}$ is along $\hat{y}$ and propagates along $\hat{z}$ → $\vec{E}$ must be along $\hat{x}$:

General expressions for phasors: $\tilde{E}(z) = \hat{x}E_{x0}e^{-\alpha z}e^{-j\beta z}$

$$\tilde{H}(z) = \frac{\hat{y}E_{x0}}{\eta_c}e^{-\alpha z}e^{-j\beta z}$$

Now we determine α, β and η_c for seawater:

Evaluate $\frac{\epsilon''}{\epsilon'}$ We have $\omega = 2\pi f = 2\pi \times 10^3 \left[\frac{rad}{s}\right]$ $f = 1$ kHz

$$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} = \frac{\sigma}{\omega\epsilon_r\epsilon_0} = \frac{4}{(2\pi \times 10^3)(80)\left(\frac{1}{36\pi} \times 10^{-9}\right)} = 9 \times 10^5 \gg 100$$

Seawater is a <u>Good</u> conductor at 1 kHz

Example: plane wave in seawater

$$\alpha = \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 10^3 \times 4\pi \times 10^{-7} \times 4} = 0.126\ [Np/m]$$

$$\beta = \alpha = 0.126\ [rad/m]$$

$$\eta_c = (1+j)\frac{\alpha}{\sigma} = \left(\sqrt{2}e^{\frac{j\pi}{4}}\right)\frac{0.126}{4} = 0.044e^{\frac{j\pi}{4}}$$

$$E_{x0} = |E_{x0}|e^{j\varphi_0}$$

$$\vec{E}(z,t) = Re\left[\hat{x}|E_{x0}|e^{j\varphi_0}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$

$$\vec{E}(z,t) = \hat{x}|E_{x0}|e^{-0.126z}\cos(2\pi 10^3 t - 0.126z + \varphi_0)\ \ [V/m]$$

$$\vec{H}(z,t) = Re\left[\frac{\hat{y}|E_{x0}|e^{j\varphi_0}}{0.044e^{\frac{j\pi}{4}}}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$

$$= \hat{y}22.5|E_{x0}|e^{-0.126z}\cos(2\pi 10^3 t - 0.126z + \varphi_0 - 45°)\ \ \left[\frac{A}{m}\right]$$

Example: plane wave in seawater

At $z = 0$: $\vec{H}(0,t) = \hat{y}22.5|E_{x0}|\cos(2\pi 10^3 t + \varphi_0 - 45°)\ \left[\frac{A}{m}\right]$

Compare with given:

$$\vec{H}(0,t) = \hat{y}100\cos(2\pi 10^3 t + 15°)\ [mA/m]$$

$$22.5|E_{x0}| = 100 \times 10^{-3} \rightarrow |E_{x0}| = 4.44\ mV/m$$

$$\varphi_0 - 45° = 15° \rightarrow \varphi_0 = 60°$$

$$\vec{E}(z,t) = \hat{x}4.44e^{-0.126z}\cos(2\pi 10^3 t - 0.126z + 60°)\ [mV/m]$$

$$\vec{H}(z,t) = \hat{y}100e^{-0.126z}\cos(2\pi 10^3 t - 0.126z + 15°)\ [mA/m]$$

Example: plane wave in seawater

b) Depth where magnitude of $\vec{E}$ is 1% of z=0

$$0.01 = e^{-0.126z}$$

$$z = \frac{\ln(0.01)}{-0.126} = 36.55\ m$$

EM Power Density

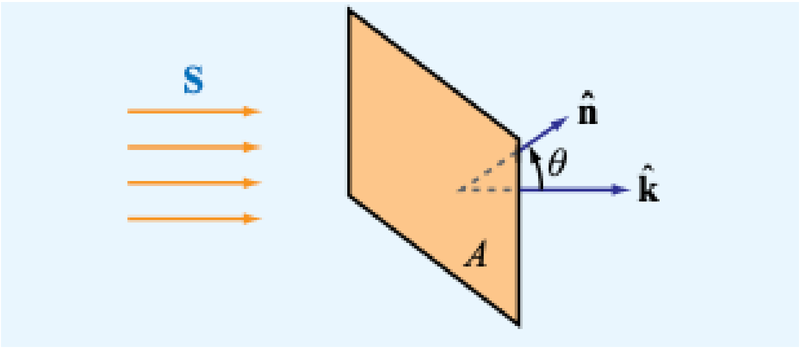
Poynting Vector: $\vec{S} = \vec{E} \times \vec{H} \quad [W/m^2]$

$$\left[\frac{V}{m}\right] \times \left[\frac{A}{m}\right] = \left[\frac{W}{m^2}\right]$$ ← Power density

Direction of Poynting vector along propagation

Total power through aperture: $P = \int_A \vec{S} \cdot \hat{n} dA \quad [W]$

Wave propagating along $\hat{k}$, angle θ with $\hat{n}$



$$P = SAcos\theta$$
$$S = |S|$$

$P(z,t) = v(z,t)i(z,t)$ -- instantaneous power

EM Power Density

$\vec{S}$ is function of time – time average power density:

$$\vec{S}_{av} = \frac{1}{2} Re\left[\tilde{E} \times \tilde{H}^*\right] \quad \left[\frac{W}{m^2}\right]$$

EM equivalent of time-average power in transmission line

$$P_{av}(z) = \frac{1}{2} Re[\tilde{V}(z)\tilde{I}^*(z)]$$

Power of plane wave in lossless medium

General plane waves in $+z$ direction:

$$\tilde{E}(z) = \hat{x}\tilde{E}_x(z) + \hat{y}\tilde{E}_y(z)$$

$$\tilde{E}(z) = (\hat{x}E_{x0} + \hat{y}E_{y0})e^{-jkz}$$

Complex in general

Magnitude of $\tilde{E}$: $\quad \left|\tilde{E}\right| = \left(\tilde{E} * \tilde{E}^*\right)^{1/2} = [|E_{x0}|^2 + |E_{y0}|^2]^{1/2}$

$$\widetilde{H}(z) = \left(\hat{x}\widetilde{H}_x + \hat{y}\widetilde{H}_y\right)e^{-jkz} = \frac{1}{\eta}\hat{z} \times \tilde{E} = \frac{1}{\eta}(-\hat{x}E_{y0} + \hat{y}E_{x0})e^{-jkz}$$

$$\vec{S}_{av} = \frac{1}{2}Re\left[\tilde{E} \times \widetilde{H}^*\right] = \hat{z}\frac{1}{2\eta}\left(|E_{x0}|^2 + \left|E_{y0}\right|^2\right) = \hat{z}\frac{\left|\tilde{E}\right|^2}{2\eta} \quad \left[\frac{W}{m^2}\right]$$

Power of plane wave in lossy medium

$$\tilde{E}(z) = \hat{x}\tilde{E}_x(z) + \hat{y}\tilde{E}_y(z) = (\hat{x}E_{x0} + \hat{y}E_{y0})e^{-\alpha z}e^{-j\beta z}$$

$$\widetilde{H}(z) = \frac{1}{\eta_c}(-\hat{x}E_{y0} + \hat{y}E_{x0})e^{-\alpha z}e^{-j\beta z}$$

$$\vec{S}_{av} = \frac{1}{2}Re\left[\tilde{E} \times \widetilde{H}^*\right] = \hat{z}\frac{1}{2}\left(|E_{x0}|^2 + \left|E_{y0}\right|^2\right)e^{-2\alpha z}Re\left[\frac{1}{\eta_c^*}\right]$$

$$\eta_c = |\eta_c|e^{j\theta_\eta}$$

Lossy medium: $$\vec{S}_{av}(z) = \frac{\hat{z}\left|\tilde{E}(0)\right|^2}{2|\eta_c|}e^{-2\alpha z}cos\theta_\eta \;\left[\frac{W}{m^2}\right]$$

Wave in lossy medium – propagates distance $z = \delta_s = \frac{1}{\alpha}$

Magnitude of E, H reduce 1/e $\approx 37\%$

Any power density decreases by $e^{-2} \approx 14\%$

Power ratio in dB

$$G = \frac{P_1}{P_2} \qquad G(dB) = 10\log(G) = 10\log\left(\frac{P_1}{P_2}\right) = 10\log(\frac{V_1^2/R}{V_2^2/R})$$

$$P_1 = \frac{V_1^2}{R}$$

$$G(dB) = 20\log\left(\frac{V_1}{V_2}\right) \rightarrow g(dB) = 20\log(g)$$

Voltage (or current) ratio scale is 20, power scale is 10

G	G [dB]
10^x	$10x$ dB
4	6 dB
2	3 dB
1	0 dB
0.5	−3 dB
0.25	−6 dB
0.1	−10 dB
10^{-3}	−30 dB

Example: power density received

Submarine at 200m depth uses antenna to receive signal transmission at 1 kHz. Determine power density onto antenna

Our prior example of EM wave propagation in seawater: $\epsilon_r = 80, \mu_r = 1, \sigma = 4\ S/m$

$$\vec{E}(z,t) = \hat{x}4.44e^{-0.126z}\cos(2\pi 10^3 t - 0.126z + 60°)\ [mV/m]$$

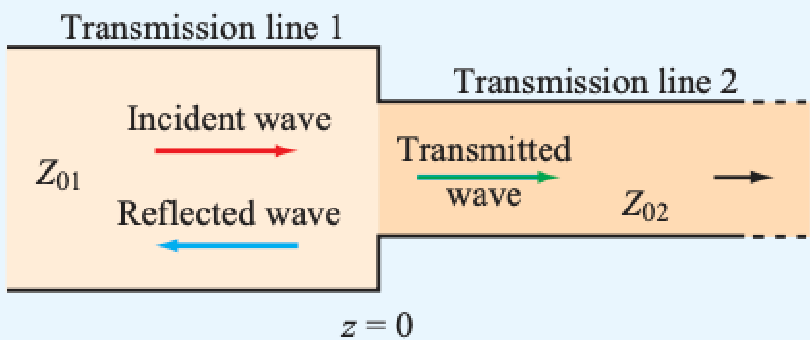
$$\left|\tilde{E}(0)\right| = |E_{x0}| = 4.44\ mV/m \qquad \alpha = 0.126\ \left[\frac{Np}{m}\right] \qquad \eta_c = 0.044\angle 45°$$

$$\vec{S}_{av}(z) = \frac{\hat{z}\left|\tilde{E}(0)\right|^2}{2|\eta_c|}e^{-2\alpha z}cos\theta_\eta = \frac{\hat{z}(4.44\times 10^{-3})^2}{2(0.044)}e^{-0.252z}cos45°$$

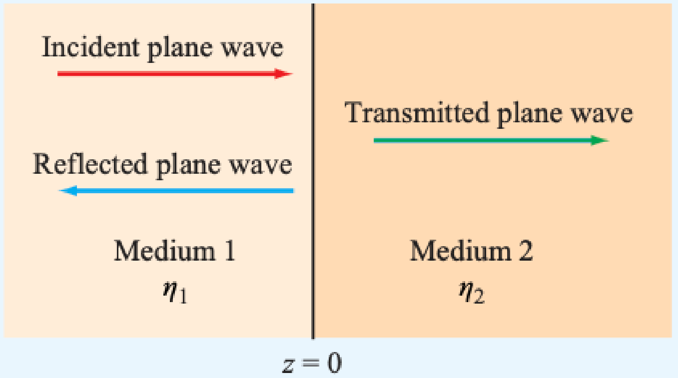
$$\vec{S}_{av}(z) = \hat{z}(0.16)e^{-0.252z}\quad \left[\frac{mW}{m^2}\right]$$

At z = 200m, $\vec{S}_{av}(z) = \hat{z}2.1\times 10^{-26}\left[\frac{W}{m^2}\right]$

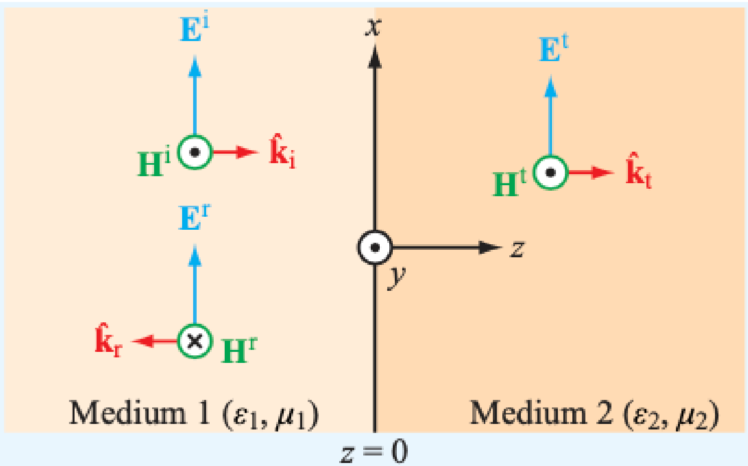
Wave reflection/transmission



(a) Boundary between transmission lines

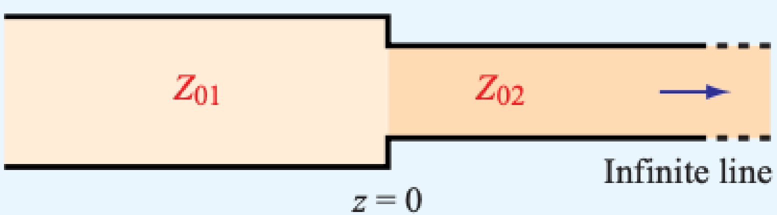


(b) Boundary between different media



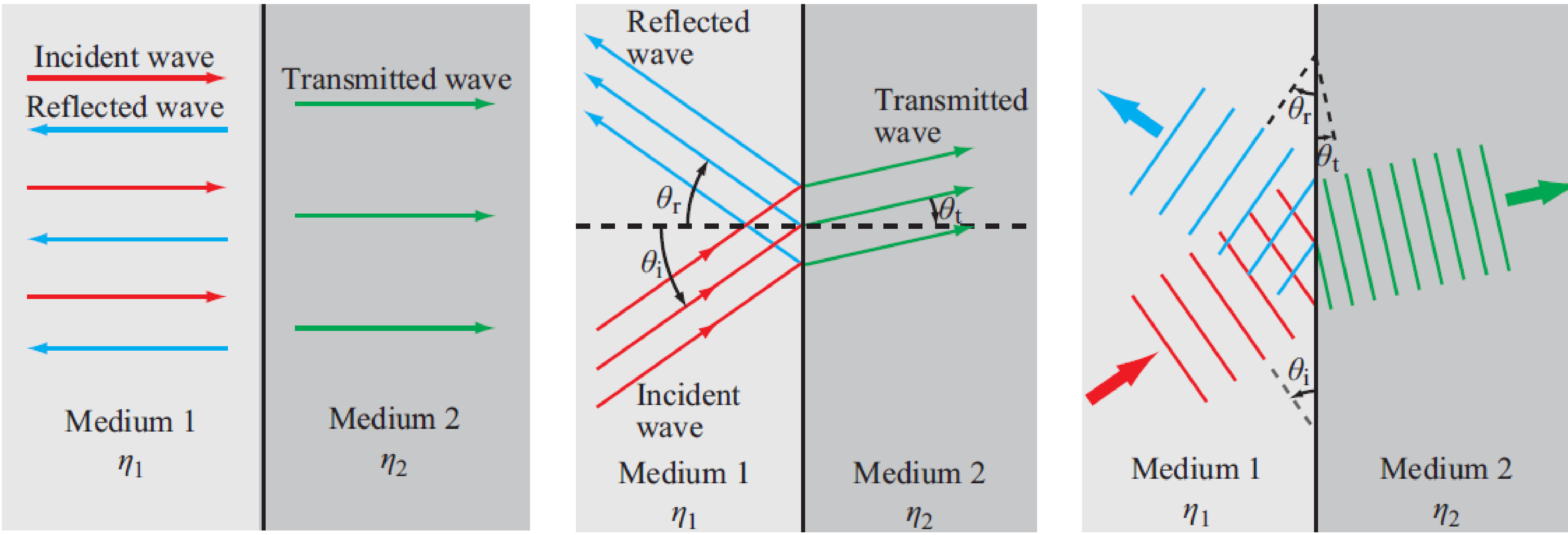
(a) Boundary between dielectric media

Analog TL:



(b) Transmission-line analogue

Wave reflection and transmission at interfaces

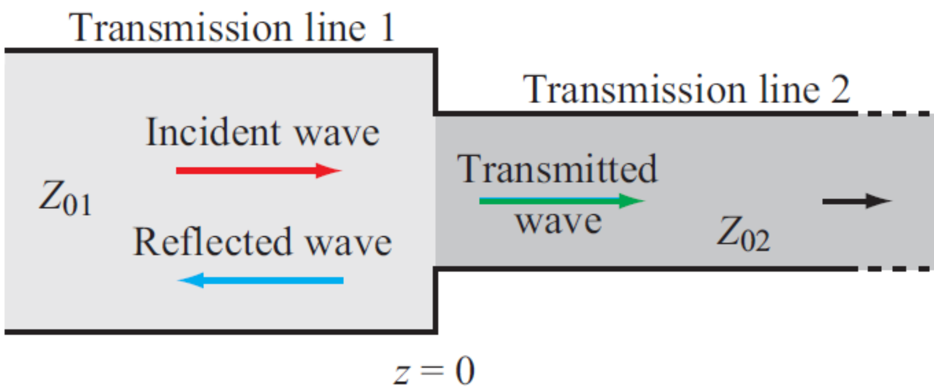


(a) Normal incidence

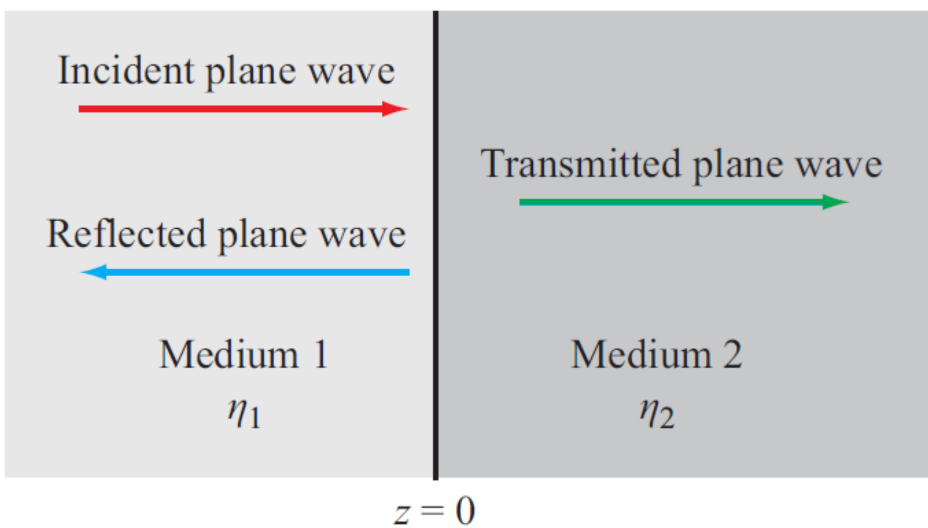
(b) Ray representation of oblique incidence

(c) Wavefront representation of oblique incidence

Normal incidence – analogy with transmission line

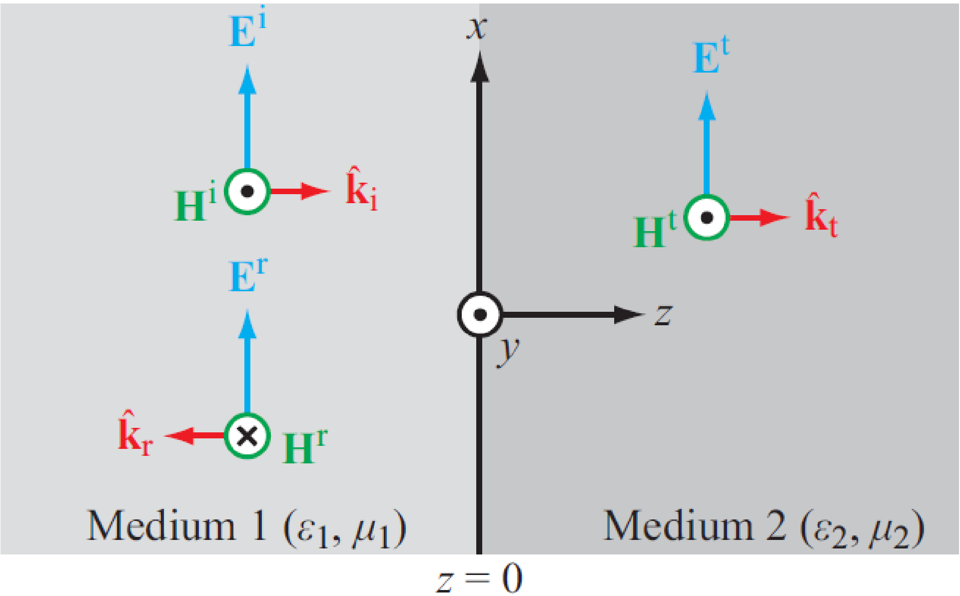


(a) Boundary between transmission lines

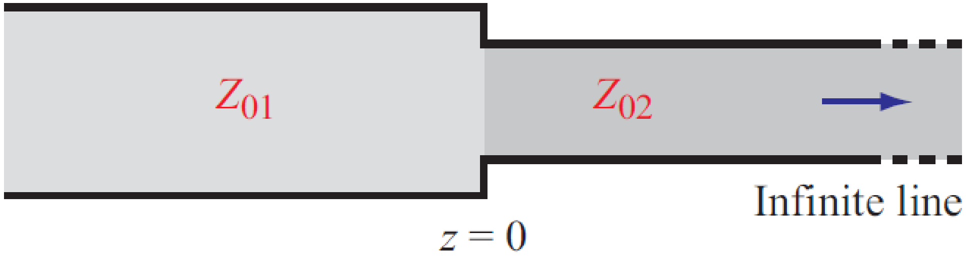


(b) Boundary between different media

Normal incidence wave reflection/transmission



(a) Boundary between dielectric media



(b) Transmission-line analogue

Wave reflection/transmission

2 media are lossless, homogeneous, dielectrics

<u>Incident wave:</u>

$$\tilde{E}^i(z) = \hat{x}E_0^i e^{-jk_1 z} \qquad \tilde{H}^i(z) = \hat{z} \times \frac{\tilde{E}^i}{\eta_1} = \frac{\hat{y}E_0^i}{\eta_1} e^{-jk_1 z}$$

<u>Reflected wave:</u>

$$\tilde{E}^r(z) = \hat{x}E_0^r e^{jk_1 z} \qquad \tilde{H}^r(z) = (-\hat{z}) \times \frac{\tilde{E}^r}{\eta_1} = \frac{-\hat{y}E_0^r}{\eta_1} e^{jk_1 z}$$

<u>Transmitted wave:</u>

$$\tilde{E}^t(z) = \hat{x}E_0^t e^{-jk_2 z} \qquad \tilde{H}^t(z) = \hat{z} \times \frac{\tilde{E}^t(z)}{\eta_2} = \frac{\hat{y}E_0^t}{\eta_2} e^{-jk_2 z}$$

$$E_0^i, E_0^r, E_0^t \rightarrow \text{amplitudes}$$

$$k_1 = \omega\sqrt{\mu_1 \epsilon_1} \qquad k_2 = \omega\sqrt{\mu_2 \epsilon_2}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \qquad \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$$

Apply boundary conditions at z = 0

<u>Medium 1</u> Incident + Reflected

$$\tilde{E}_1(z) = \tilde{E}^i(z) + \tilde{E}^r(z)$$

$$\tilde{E}_1(z) = \hat{x}(E_0^i e^{-jk_1 z} + E_0^r e^{jk_1 z})$$

$$\widetilde{H}_1(z) = \widetilde{H}^i(z) + \widetilde{H}^r(z)$$

$$\widetilde{H}_1(z) = \hat{y}\frac{1}{\eta_1}(E_0^i e^{-jk_1 z} - E_0^r e^{jk_1 z})$$

<u>Medium 2</u> Transmitted

$$\tilde{E}_2(z) = \tilde{E}^t(z) = \hat{x}E_0^t e^{-jk_2 z}$$

$$\widetilde{H}_2(z) = \widetilde{H}^t(z) = \frac{\hat{y}E_0^t}{\eta_2}e^{-jk_2 z}$$

Boundary conditions at $z = 0$: Tangential components of E, H continuous

$$\tilde{E}_1(0) = \tilde{E}_2(0) \rightarrow E_0^i + E_0^r = E_0^t$$

$$\widetilde{H}_1(0) = \widetilde{H}_2(0) \rightarrow \frac{E_0^i}{\eta_1} - \frac{E_0^r}{\eta_1} = \frac{E_0^t}{\eta_2}$$

$$E_0^r = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)E_0^i = \Gamma E_0^i \qquad E_0^t = \left(\frac{2\eta_2}{\eta_2 + \eta_1}\right)E_0^i = \tau E_0^i$$

Normal Incidence

$$\Gamma = \frac{E_0^r}{E_0^i} = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)$$ Reflection coefficient

$$\tau = \frac{E_0^t}{E_0^i} = \left(\frac{2\eta_2}{\eta_2 + \eta_1}\right)$$ Transmission coefficient

Similar form as for transmission lines

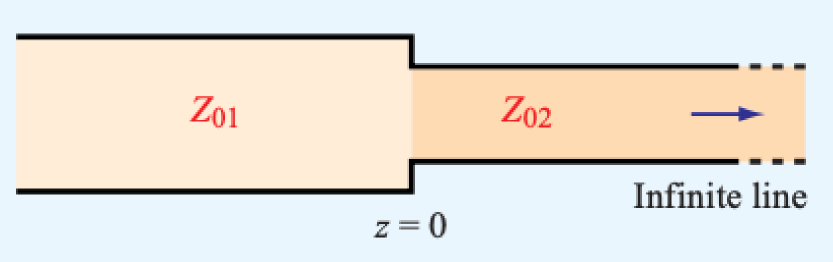
Γ, τ may be complex for conductive media

$$\tau = 1 + \Gamma \qquad \text{(normal incidence)}$$

For non-magnetic media, $\eta_1 = \frac{\eta_0}{\sqrt{\epsilon_{r1}}}$ $\eta_2 = \frac{\eta_0}{\sqrt{\epsilon_{r2}}}$ Free space impedance

$$\Gamma = \frac{\sqrt{\varepsilon_{\mathrm{r}_1}} - \sqrt{\varepsilon_{\mathrm{r}_2}}}{\sqrt{\varepsilon_{\mathrm{r}_1}} + \sqrt{\varepsilon_{\mathrm{r}_2}}} \qquad \text{(nonmagnetic media).}$$

Transmission Line Analogue



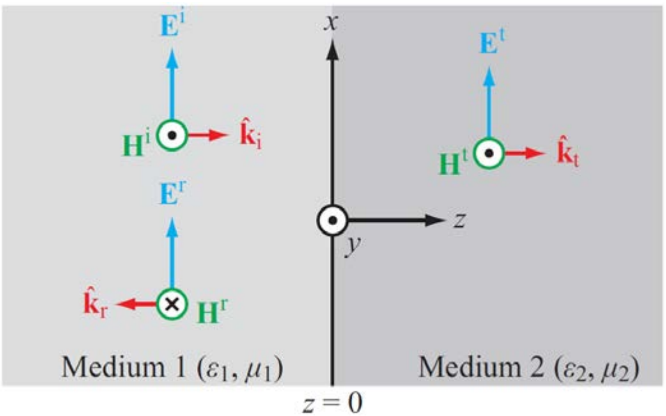
(b) Transmission-line analogue

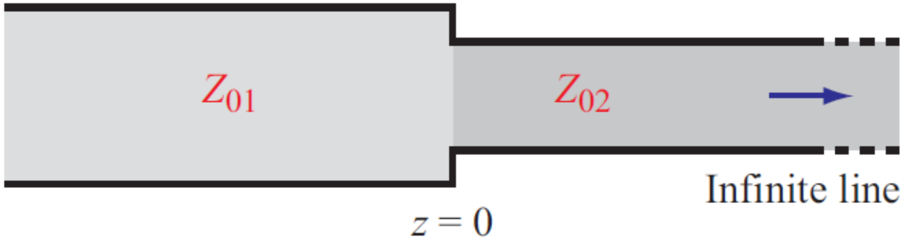
From TL: $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

For infinite line lossless TL input impedance is the characteristic impedance:

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \qquad \tau = 1 + \Gamma$$

Transmission Line Analogue – normal incidence





$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \qquad \tau = 1 + \Gamma$$

$$\beta_1 = \omega\sqrt{\mu_1\epsilon_1} \qquad \beta_2 = \omega\sqrt{\mu_2\epsilon_2}$$

ω

$$\widetilde{\boldsymbol{E}}_{\mathbf{1}}(z) = \hat{x}E^i_0(e^{-jk_1z} + \Gamma e^{jk_1z})$$

$$\widetilde{V}_1(z) = V_0{}^+(e^{-j\beta_1z} + \Gamma e^{j\beta_1z})$$

$$\widetilde{\boldsymbol{H}}_{\mathbf{1}}(z) = \hat{y}\frac{E^i_0}{\eta_1}(e^{-jk_1z} - \Gamma\ \ e^{jk_1z})$$

$$\tilde{I}_1(z) = \frac{V_0{}^+}{Z_{01}}(e^{-j\beta_1z} - \Gamma e^{j\beta_1z})$$

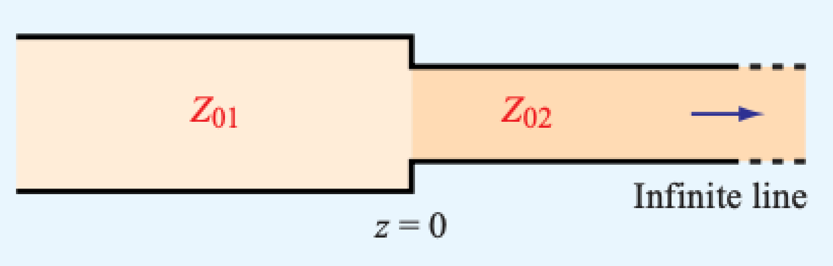
$$\widetilde{\boldsymbol{E}}_{\mathbf{2}}(z) = \hat{x}\tau E^i_0 e^{-jk_2z}$$

$$\widetilde{V}_2(z) = \tau V_0{}^+(e^{-j\beta_2z})$$

$$\widetilde{\boldsymbol{H}}_{\mathbf{2}}(z) = \hat{y}\tau\ \frac{E^i_0}{\eta_2}e^{-jk_2z}$$

$$\tilde{I}_2(z) = \frac{\tau V_0{}^+}{Z_{02}}(e^{-j\beta_2z})$$

Transmission Line Analogue



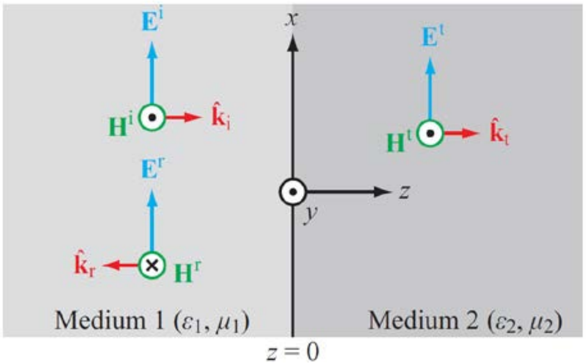
(b) Transmission-line analogue

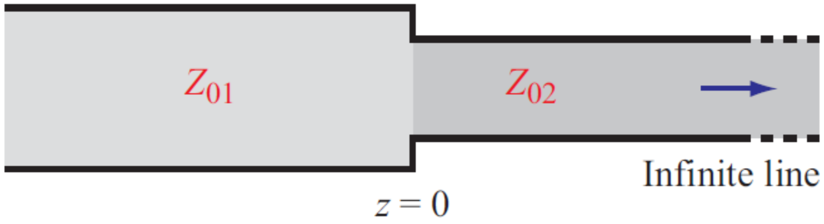
Standing wave ratio: incident + reflected waves $$S = \frac{|\tilde{E}_1|_{max}}{|\tilde{E}_1|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

If 2 media have equal $\eta_1 = \eta_2$ $\Gamma = 0, S = 1$

If medium #2 is perfect conductor: $\eta_2 = 0$ $\Gamma = -1, S = \infty$

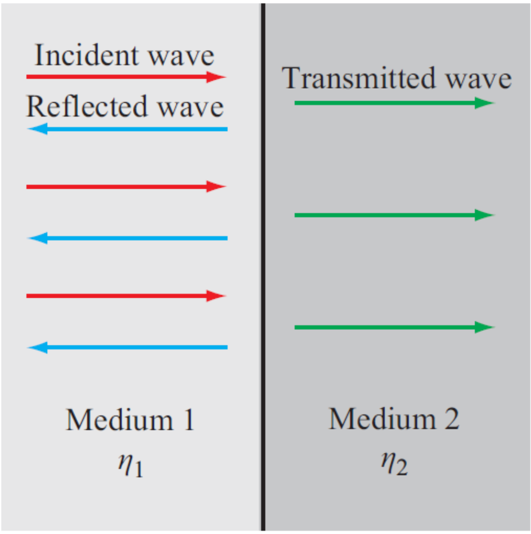
Transmission Line Analogue - summary





Plane Wave [Fig. 8-4(a)]	Transmission Line [Fig. 8-4(b)]
$\widetilde{\mathbf{E}}_1(z) = \hat{\mathbf{x}} E_0^i (e^{-jk_1 z} + \Gamma e^{jk_1 z})$ (8.5a)	$\widetilde{V}_1(z) = V_0^+ (e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z})$ (8.5b)
$\widetilde{\mathbf{H}}_1(z) = \hat{\mathbf{y}} \dfrac{E_0^i}{\eta_1} (e^{-jk_1 z} - \Gamma e^{jk_1 z})$ (8.6a)	$\widetilde{I}_1(z) = \dfrac{V_0^+}{Z_{01}} (e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z})$ (8.6b)
$\widetilde{\mathbf{E}}_2(z) = \hat{\mathbf{x}} \tau E_0^i e^{-jk_2 z}$ (8.7a)	$\widetilde{V}_2(z) = \tau V_0^+ e^{-j\beta_2 z}$ (8.7b)
$\widetilde{\mathbf{H}}_2(z) = \hat{\mathbf{y}} \tau \dfrac{E_0^i}{\eta_2} e^{-jk_2 z}$ (8.8a)	$\widetilde{I}_2(z) = \tau \dfrac{V_0^+}{Z_{02}} e^{-j\beta_2 z}$ (8.8b)
$\Gamma = (\eta_2 - \eta_1)/(\eta_2 + \eta_1)$	$\Gamma = (Z_{02} - Z_{01})/(Z_{02} + Z_{01})$
$\tau = 1 + \Gamma$	$\tau = 1 + \Gamma$
$k_1 = \omega\sqrt{\mu_1 \varepsilon_1}$, $k_2 = \omega\sqrt{\mu_2 \varepsilon_2}$	$\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1}$, $\beta_2 = \omega\sqrt{\mu_2 \varepsilon_2}$
$\eta_1 = \sqrt{\mu_1/\varepsilon_1}$, $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$	Z_{01} and Z_{02} depend on transmission-line parameters

Power Flow



Medium 1: $\vec{S}_{av_1}(z) = \frac{1}{2} Re\left[\tilde{E}_1(z) \times \widetilde{H}_1^*(z)\right]$

$$\vec{S}_{av_1}(z) = \frac{1}{2} Re\left[\hat{x} E^i_0 (e^{-jk_1 z} + \Gamma e^{jk_1 z}) \times \hat{y} \frac{{E^i_0}^*}{\eta_1} (e^{jk_1 z} - \Gamma^* e^{-jk_1 z})\right]$$

$$\vec{S}_{av_1}(z) = \hat{z} \frac{|E^i_0|^2}{2\eta_1} (1 - |\Gamma|^2)$$

$$\vec{S}_{av_1} = \vec{S}^i_{av} + \vec{S}^r_{av} \quad \left\{ \begin{aligned} \vec{S}^i_{av} &= \hat{z} \frac{|E^i_0|^2}{2\eta_1} \\ \vec{S}^r_{av} &= -\hat{z}\, |\Gamma|^2 \frac{|E^i_0|^2}{2\eta_1} \end{aligned} \right\} \quad \vec{S}^r_{av} = -\,|\Gamma|^2 \vec{S}^i_{av}$$

Power Flow

Medium 2:

$\tau = 1 + \Gamma$ $\qquad$ $\Gamma \rightarrow$ real for lossless, can be complex for conducting

$$\vec{S}_{av_2}(z) = \frac{1}{2} Re\left[\hat{x}\tau E_0^i e^{-jk_2 z} \times \hat{y}\tau^* \frac{E_0^{i*}}{\eta_2} e^{jk_2 z}\right] = \hat{z}|\tau|^2 \frac{|E_0^i|^2}{2\eta_2}$$

$$\vec{S}_{av_1}(z) = \hat{z}\frac{|E_0^i|^2}{2\eta_1}(1 - |\Gamma|^2)$$

$$\frac{\tau^2}{\eta_2} = \frac{1 - \Gamma^2}{\eta_1} \qquad \text{(lossless media),}$$

leads to

$$\mathbf{S}_{\mathrm{av}_1} = \mathbf{S}_{\mathrm{av}_2}.$$